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A STRUCTURAL OPTIMIZATION TECHNIQUE

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Sensitivity analysis is performed for general structures. Gradual improvement in the design of a vibrating structure is attempted by an iterative process, where the frequency of vibration depends upon the design. Abstract Fréchet differentiation is used to establish necessary criteria for optimization. Particular attention is given to the problem of optimization of gun tube.

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0. Introductory remarks

Design of vibrating structures has been carried out on trial and error basis. No comprehensive theory exists at the present time, although some progress in optimizing the design of a single element has been made. Prager and his collaborators worked primarily on optimization of the design of a single beam, making use of the Betti-Castigliano deflection formula and deriving some theoretical results for optimization of weight for a given deflection at a known point along the length of the beam, and related problems (see [7] or [8]). A more difficult problem of simultaneous optimal design and optimal control was considered by Komkov and Coleman in [4], however the applications of their work remain limited, and no general optimization theory was derived.

A different static optimization was pursued by Haug [3], Haug et al [2], and others, who attempted to derive algorithms for gradual improvement in design by an iterative process. So far the numerical results obtained at the University of Iowa are promising, and have been applied to some problems of weapon design.

The purpose of this work is to generalize the results of [3] via sensitivity analysis. In particular to derive sensitivity (to design changes) for beams, plates and general structures, and to translate these theoretical results into a computational algorithm using some form of the steepest descent algorithm.

A particular application of the technique derived by Komkov and Coleman ([4]) is given to the problem of a gun tube design.

1. A class of optimal design problems described by a distributed parameter system.

Consider a design problem postulated on a subset $\Omega \subseteq \mathbb{R}^k$, with a local coordinate system $\tilde{x} = (x_1, x_2, \ldots, x_k)$. The state of the system is determined by an n-dimensional vector function $\tilde{z}(\tilde{x})$, which is an element of a Hilbert space H_1.

The design of the system is described by an m-dimensional vector function $\tilde{u}(\tilde{x})$. There are constraints on stress, deflection, etc., and physical constrains on the design vector $\tilde{u}(\tilde{x})$. The set of m-dimensional vector functions satisfying the constraints will be called the set of admissible designs and will be denoted by $U(\tilde{x})$.

The state vector $\tilde{z}$ satisfies a system of differential equations

$$L(\tilde{u})\tilde{z} = Q(\tilde{x},\tilde{u}), \quad \tilde{x} \in \Omega, \tag{1}$$

where the forcing function Q is an element of a Hilbert space H_3. Thus $L\colon\ H_1 \to H_3$. The state variable $\tilde{z}$ also satisfies a set of boundary conditions

$$B\ \tilde{z}(\tilde{x}) = \tilde{q}(\tilde{x}), \quad \tilde{x} \in \partial\Omega, \tag{2}$$

Introduce also Hilbert spaces H_2, H_4 of functions whose domain is $\partial\Omega$, and such that the boundary condition (2) is given as a mapping $B\colon\ H_2 \to H_4$.

While the same symbol $\tilde{z}$ is used in relations (1) and (2) it

actually depicts different classes of functions. However, $\underset{\sim}{z}$ regarded as an element of $H_1 \oplus H_2$ is continuous on $\Omega \cup \partial\Omega$ and no problems will arise due to the simplified notation.

The system (1) and (2) will be referred as B.V.P. (boundary-value problem).

Let A denote the operator

$$A = \begin{bmatrix} L \\ B \end{bmatrix} : \; H_1 \oplus H_2 \to H_3 \oplus H_4 .$$

The existence of a unique solution of B.V.P. is equivalent to the statement that A^{-1} is defined. Well posedness of the problem, in the sense of Hadamard, is equivalent to the statement that A^{-1} is defined and is a bounded operator from $H_3 \oplus H_4$ into $H_1 \oplus H_2$. This is a well known property of stable structural designs. In an important class of structural designs, the differential operator L also turns out to be positive definite and bounded below (away from zero), and L^{-1} is a completely continuous operator.

Also of interest in this paper are designs of structural members subjected to buckling and to natural frequency constraints.

In each of these cases, the physical problem leads to an eigenvalue system

$$L'(\underset{\sim}{u})\, \underset{\sim}{y}(\underset{\sim}{x}) = \zeta M(\underset{\sim}{u})\, \underset{\sim}{y}(\underset{\sim}{x}) \quad , \; \underset{\sim}{x} \in \Omega, \tag{3}$$

$$B'\, \underset{\sim}{y}(\underset{\sim}{x}) = 0 \quad , \; \underset{\sim}{x} \in \partial\Omega, \tag{4}$$

where L' and M are symmetric, positive, bounded below differential

operators, ζ is a real eigenvalue, and $\underset{\sim}{y}(\underset{\sim}{x})$ is the corresponding eigenfunction. The eigenfunctions $\underset{\sim}{y}(\underset{\sim}{x})$ are normalized by the condition

$$\langle M\underset{\sim}{y},\underset{\sim}{y}\rangle = |||\underset{\sim}{y}|||^2 = 1,$$

where $|||\cdot|||$ denotes the energy norm induced by the inner product $[\underset{\sim}{X},\underset{\sim}{Y}] = \langle MX,Y\rangle$. The inner product $\langle\underset{\sim}{X},\underset{\sim}{Y}\rangle$ is defined in the usual way, i.e. $\langle\underset{\sim}{X},\underset{\sim}{Y}\rangle = \int_{\Omega}\underset{\sim}{X}(\underset{\sim}{x})\ \underset{\sim}{Y}(\underset{\sim}{x})\ d\underset{\sim}{x}$.

One wishes to minimize, or approximately minimize, a functional

$$J : \{H_1 \oplus H_2 \times U\} \to \mathbb{R}.$$

Where J is usually either an L_2 norm of Y, an energy norm, or a possibly non-linear, but positive functional. Analytically, J is of the form

$$J(\zeta,\ z(\underset{\sim}{u}),\underset{\sim}{u}) = h_0(\zeta) + \int_{\partial\Omega} g_0(\underset{\sim}{z},\underset{\sim}{x})d\underset{\sim}{x} + \int_{\Omega} f_0(\underset{\sim}{z}(\underset{\sim}{u}),\underset{\sim}{u},\underset{\sim}{x})d\underset{\sim}{x}. \quad (5)$$

The design problem may thus be reduced to selection of an admissible design $\underset{\sim}{u}(\underset{\sim}{x}) \in U$ to minimize (or approximately minimize) J, subject to the constraints (1) - (4) and performance constraints of the form

$$\psi_\alpha = h_\alpha(\zeta) + \int_{\partial\Omega} g_\alpha(\underset{\sim}{z},\underset{\sim}{x},\underset{\sim}{u})d\underset{\sim}{x} + \int_{\Omega} f_\alpha(\underset{\sim}{z},\underset{\sim}{x},\underset{\sim}{u})d\underset{\sim}{x} \begin{cases} =0, & \alpha=1,2,\ldots r^1, \quad (6\underline{a}) \\ \leq 0, & \alpha=r^1+1,\ldots,r \quad (6\underline{b}) \end{cases}$$

$$\phi_\beta(\underset{\sim}{x},\underset{\sim}{u}) \begin{cases} =0, \ \beta=1,2,\ldots\xi^1, & \\ \leq 0, \ \beta=\xi^1+1,\ldots,\xi. & \end{cases} \left\{ \underset{\sim}{x} \ \varepsilon \ \Omega \right. \qquad \begin{matrix} (7\underline{a}) \\ (7\underline{b}) \end{matrix}$$

2. Comments based on computing experience.

The pointwise constraints (7a) and (7b) are allowed to depend only on the design variable $\underset{\sim}{u}$. If the state variable $\underset{\sim}{z}$ were present, computational difficulties would arise later in the development.

Constraints of the form $\eta(\underset{\sim}{x},\underset{\sim}{u},\underset{\sim}{z}) \leq 0$, $\underset{\sim}{x} \ \varepsilon \ \Omega$, arise often, however, and must be treated. It is readily seen that the pointwise constraint $\eta(\underset{\sim}{x}) \leq 0$ is equivalent to the functional constraint

$$\int_\Omega (\eta(\underset{\sim}{x}) + |\eta(\underset{\sim}{x})|) \, d\underset{\sim}{x} = 0. \tag{8}$$

Similarly, $\phi(\underset{\sim}{x}) \leq a_0 \ \forall \underset{\sim}{x} \ \varepsilon \ \Omega$ is equivalent to

$$\int_\Omega \{(\phi(\underset{\sim}{x}) - a_0) + |\phi(\underset{\sim}{x}) - a_0|\} \, d\underset{\sim}{x} = 0.$$

3. Computation of Gateaux derivatives associated with an optimal design problem.

The standard definition (see [6]) of a Gateaux derivative of a function f: $B_1 \to B_2$ in the direction of a vector h is employed. Here B_1, B_2 are Banach spaces, $h \ \varepsilon \ B_1$. If for a sufficiently small value of the real parameter t

$$f(x_0+th) - f(x_0) = t \, L_{x_0} + r(x_0,th), \tag{9}$$

where

$$\lim_{t\to 0} \left\| \frac{r(x_0,th)}{t} \right\| = 0 \quad , \tag{9a}$$

and where L_{x_0} is a linear operator $L_{x_0}: B_1 \to B_2$, then f is Gateaux differentiable at x_0 (ϵB_1), and the linear operator L_{x_0} is called the Gateaux derivative of f in the direction of h. If

$$\lim_{t\to 0} \left\| \frac{r(x_0,th)}{t} \right\| = 0$$

uniformly with respect to all $h \in B_1$ on all bounded subsets of B_2, and if L_{x_0} is continuous, then f is called Fréchet differentiable at x_0.

Note: the left hand side of equality (9) will be called the Gateaux variation of f in the direction h, computed at x_0.

In this paper B_1 and B_2 are Hilbert spaces and, in the development that follows, B_1 will be identified with the Sobolëv space H^k, while B_2 will, in most cases, be the real line.

The minimization problem consists of finding a minimum of a functional J which in many problems of structural analysis is a quadratic functional. L_2 norm of the state variable z, an energy norm, or possibly other norms, subject to a constraint imposed on the state variable, with which must satisfy the appropriate differential equations of mathematical physics. That is, the problem is formulated as minimization problem for the functional $J(u,z,\zeta,x)$ subject to differential equations $Lz = \zeta z$. For example

$Lz = \zeta z$ may be of the form $\frac{d^2}{dx^2}[EI(x)\ \frac{d^2 z}{dx^2}] + \zeta z = 0$. Then the minimization problem may be restated in an enlarged space as that of finding the minimum

$$\Lambda_1\ [\langle EI(x)\ \frac{d^2 z}{dx^2}\ ,\ \frac{d^2 z}{dx^2}\rangle + \langle \zeta z, z\rangle] + \Lambda_0 J.$$

where Λ_0, Λ_1 are Lagrangian multipliers.

Hence the constraint equation is regarded as one of the Euler-Lagrange equations which, subject to some smoothness assumptions on the state variable z, is a necessary condition for the solution of the minimum problem in the enlarged space.

Consider the functional $J(\underset{\sim}{u},\underset{\sim}{z},\underset{\sim}{x},\zeta)$ defined by (5) and its Gateaux variation δJ, where $\delta h = th$ (for sufficiently small (t) and $\delta h = (\delta z,\ \delta u,\ \delta\zeta)$. In a purely formal fashion one may write

$$\delta J = \frac{\partial h_0}{\partial \zeta}\ \delta\zeta + \left\langle \frac{\partial g_0}{\partial \underset{\sim}{z}},\ \delta \underset{\sim}{z} \right\rangle_{\partial\Omega} + \left\langle \frac{\partial f_0}{\partial \underset{\sim}{z}},\ \delta \underset{\sim}{z} \right\rangle_{\Omega} + \left\langle \frac{\partial f_0}{\partial \underset{\sim}{u}},\ \delta \underset{\sim}{u} \right\rangle_{\Omega} + r(\underset{\sim}{h}), \tag{10}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ and $\langle\cdot,\cdot\rangle_{\Omega}$ are the appropriate inner products of functions whose domains are $\partial\Omega$ and Ω, respectively, and

$$\lim_{\|\underset{\sim}{h}\|\to 0} \frac{\|r(\underset{\sim}{h})\|}{\|\underset{\sim}{h}\|} = 0.$$

The problem arises of justifying this formal approach by showing differentiability of the operator L' with respect to the eigenvalue ζ. There are some obvious difficulties, since L' is an unbounded operator. Instead, the fact that A^{-1}, defined by 1a, is

a compact operator in common problems of structural analysis, can be employed. Hence, the eigenvalues of A^{-1}, which are ζ^{-1}, are a discrete subset of the real line, i.e. the spectrum of A^{-1} consists only of isolated eigenvalues. One may now involk the theorem on bounded invertibility (see Kato [9] pp. 196-197) and a well known result on perturbation of spectrum of A^{-1}; namely that the spectrum of A^{-1} changes continuously with A^{-1}, such small changes being interpreted in the usual operator norm topology ([9]).

For example, consider a beam equation of the form

$$\left[\frac{d^2}{dx^2}\sqrt{EI}\left(\sqrt{EI}\,\frac{d^2}{dx^2}\right)\right] y + q_n(x)y = f(x) + \xi_n y \tag{11}$$

subject to appropriate boundary conditions. Let $q_n \xrightarrow{L_2[0,1]} q$, $f \in L_2[0,1]$. Then $R(\xi_n) \to R(\xi)$, where R is the resolvent, and $\lim_{n\to\infty} \xi_n = \xi$, and ξ does not lie in the convex hull of ξ_n implies that the solutions of (11) converge in $L_2[0,1]$ to the solution of the equation

$$\left[\frac{d^2}{dx^2}\sqrt{EI}\left(\sqrt{EI}\,\frac{d^2}{dx^2}\right)\right] y + q(x)y = f(x) + \xi y. \tag{12}$$

The key in this discussion is the avoidance of singular perturbations.

To illustrate this remark consider the following two examples. The fourth order beam operator

$$\frac{d^2}{dx^2}\left(EI(x)\,\frac{d^2}{dx^2}\right), \tag{13}$$

with boundary conditions of either free support at both ends, or cantelevered at one end, implies continuous changes in the eigenvalues with a continuous change in EI(x). The operator

$$\left(\frac{d^2}{dx^2} EI(x) \frac{d^2}{dx^2} - \alpha \frac{d^2}{dx^2} \right) \tag{14}$$

with $\begin{cases} y(0) = y'(0) = 0 \\ y(1) = y'(1) = 0, \end{cases}$

on the other hand, will cause singular perturbations. (Note: This is exactly the example of singular perturbation given in [9] page 436, example 1.20).

That is, the eigenvalues of (13) are stable and of (14) are not.

Note: The discussion of stability of eigenvalues can be traced to Lord Rayleigh (see [10], page 300).

It is assumed in the use of formula (10) that A^{-1} is compact and that perturbations of eigenvalues of A^{-1} (and of A) are stable. That is, one is dealing with a common case of stable structural design.

4. Manipulation of formula (10).

The meaning of $\delta\zeta$ and its relationship to δu needs to be determined next in the interpretation of formula (10). Regarding $\delta\zeta$ as the Gateaux variation of the first eigenvalue of the system $L'y = \zeta My$, and assuming that L' and M depend holomorphically on $\underset{\sim}{u}$, it is possible to apply Aronszajn's theory of general quadratic forms and justify formal differentiation of the associated Rayleigh

quotient. (see Kato [9], VII - 83, pp. 421-422).

The first eigenvalue of the system is equal to the value of the Rayleigh quotient

$$\zeta_1 = \frac{\langle L'\underset{\sim}{y}_1, \underset{\sim}{y}_1\rangle_\Omega}{\langle M\underset{\sim}{y}_1, \underset{\sim}{y}_1\rangle_\Omega},$$

where $\underset{\sim}{y}_1(x)$ is the corresponding eigenfunction. Assuming symmetry of L' and M and positive definiteness of M, and using formal rules of Gateaux or Fréchet differentiation (see for example the introductory notes in [11]) for the Rayleigh functional, one obtains

$$\zeta_{1_h} = \langle M\underset{\sim}{y}_1, \underset{\sim}{y}_1\rangle_\Omega^{-1} \{\langle (L'\underset{\sim}{y}_1)_h, \underset{\sim}{y}_1\rangle_\Omega + \langle L'\underset{\sim}{y}_1, \underset{\sim}{y}_{1_h}\rangle_\Omega - \langle L'\underset{\sim}{y}_1, \underset{\sim}{y}_1\rangle$$

$$(\langle M\underset{\sim}{y}_1, \underset{\sim}{y}_1\rangle_\Omega)^{-1} [\langle (M\underset{\sim}{y}_1)_h, \underset{\sim}{y}_1\rangle_\Omega + \langle M\underset{\sim}{y}_1, \underset{\sim}{y}_{1_h}\rangle_\Omega]\}$$

$$= \langle M\underset{\sim}{y}_1, \underset{\sim}{y}_1\rangle_\Omega^{-1} \{\langle (L'\underset{\sim}{y}_1)_h, \underset{\sim}{y}_1\rangle - \zeta\langle (M\underset{\sim}{y}_1)_h, \underset{\sim}{y}_1\rangle + \langle (L'\underset{\sim}{y}_1 -$$

$$\zeta M\underset{\sim}{y}_1), \underset{\sim}{y}_{1_h}\rangle\}, \tag{15}$$

where subscript h denotes Fréchet differentiation with respect to a typical component h of the vector $\underset{\sim}{u}$.

Expressions of the type $(L'\underset{\sim}{y})_h$ have to be carefully interpreted, since $\underset{\sim}{y}_1$ and the map L' both depend on h. If both are Fréchet differentiable, the following rule is easily justified

$$\delta(L\underset{\sim}{y}) = (\delta L)\underset{\sim}{y} + L\delta\underset{\sim}{y} \tag{16}$$

or equivalently

$$(L\underset{\sim}{y})_h = L_h\underset{\sim}{y} + L\underset{\sim}{y}_h$$

where $L_h y$ has to be interpreted as the Gateaux variation of the operator L with y regarded as fixed, while y_h is the corresponding variation of y.

Observe: $L(h_0+th)\underset{\sim}{y}(h_0+th)-L(h_0)\underset{\sim}{y}(h_0) = [L(h_0+th)\underset{\sim}{y}(h_0+th)-L(h_0)\underset{\sim}{y}(h_0+th)]$ $+ [L(h_0)\underset{\sim}{y}(h_0+th)-L(h_0)\underset{\sim}{y}(h_0)]$.

The Gateaux variation of the first eigenvalue $\delta\zeta_1$ is given by

$$\delta\zeta_1(u) = \left\langle M\underset{\sim}{y}_1, \underset{\sim}{y}_1\right\rangle_\Omega^{-1} \left\{\left\langle\frac{\partial(L'\underset{\sim}{y}_1)}{\partial \underset{\sim}{u}}, \underset{\sim}{y}_1\right\rangle_\Omega - \zeta\left\langle\frac{\partial(M\underset{\sim}{y}_1)}{\partial \underset{\sim}{u}}, \underset{\sim}{y}_1\right\rangle_\Omega \delta\underset{\sim}{u}.\right.$$

$$\left. + \left\langle(L'\underset{\sim}{y}_1 - \delta M\underset{\sim}{y}_1), \frac{\partial \underset{\sim}{y}_1}{\partial u}\right\rangle_\Omega\right\} \delta\underset{\sim}{u}.$$

Since $\underset{\sim}{y}_1$ is the eigenfunction of (3) the last term can be omitted, and

$$\delta\zeta_1 = \left\langle M\underset{\sim}{y}_1, \underset{\sim}{y}_1\right\rangle^{-1} \left\{\left\langle\frac{\partial(L'\underset{\sim}{y}_1)}{\partial u}, \underset{\sim}{y}_1\right\rangle_\Omega - \zeta\left\langle\frac{\partial(M\underset{\sim}{y}_1)}{\partial u}, \underset{\sim}{y}_1\right\rangle\right\} \delta\underset{\sim}{u}. \tag{17}$$

<u>Note</u>: $\left\langle\frac{\partial L'\underset{\sim}{y}_1}{\partial u}, \underset{\sim}{y}_1\right\rangle_\Omega$ is not a scalar, it is a vector in the space dual to $\underset{\sim}{u}$. The same remark applies to the vector $\left\langle\frac{\partial M\underset{\sim}{y}_1}{\partial u}, \underset{\sim}{y}_1\right\rangle_\Omega$.

5. An adjoint problem.

Since the generalized displacement (or state) vector $\underset{\sim}{z}$ is an element of the Hilbert space $H_1 \oplus H_2$, its dual (the generalized force) is an element of the same Hilbert space. The duality is maintained for sake of physical interpretation.

Let $\underset{\sim}{\lambda}^J$ be an element of the dual space, satisfying the differential equation

$$L^* \underset{\sim}{\lambda}^J = \frac{\partial f_0}{\partial \underset{\sim}{z}} \quad \text{in } \Omega , \tag{18}$$

where L* is a formal adjoint of L, satisfying

$$\left\langle \underset{\sim}{W}, L\underset{\sim}{z} \right\rangle_\Omega = \left\langle L^*\underset{\sim}{W}, \underset{\sim}{z} \right\rangle_\Omega + \left\langle C\underset{\sim}{W}, \underset{\sim}{z} \right\rangle_{\partial\Omega} , \tag{19}$$

where C is a linear operator C: $H_2 \to H_2$.

$\underset{\sim}{\lambda}^J$ satisfies boundary condition

$$\left\langle (C\underset{\sim}{\lambda}^J - \frac{\partial g_0}{\partial \underset{\sim}{z}}), \delta z \right\rangle_{\partial\Omega} = 0, \tag{20}$$

for every admissible $\delta \underset{\sim}{z}$, which is certainly satisfied if

$$C\underset{\sim}{\lambda}^J = \frac{\partial g_0}{\partial \underset{\sim}{z}} \quad \text{on } \partial\Omega . \tag{21}$$

Hence

$$\left\langle L\delta \underset{\sim}{z}, \underset{\sim}{\lambda}^J \right\rangle_\Omega = \left\langle \frac{\partial f_0}{\partial \underset{\sim}{z}} , \delta \underset{\sim}{z} \right\rangle_\Omega + \left\langle C\underset{\sim}{\lambda}^J, \delta \underset{\sim}{z} \right\rangle_{\partial\Omega}$$

and

$$\delta J = \frac{\partial h_0}{\partial \zeta}\,\delta\zeta + \left\langle \frac{\partial g_0}{\partial \underset{\sim}{z}},\ \delta\underset{\sim}{z} \right\rangle_{\partial\Omega} + \left\langle \frac{\partial f_0}{\partial \underset{\sim}{z}},\ \delta\underset{\sim}{z} \right\rangle_{\Omega} + \left\langle \frac{\partial f_0}{\partial \underset{\sim}{u}},\ \delta\underset{\sim}{u} \right\rangle_{\Omega}$$

$$= \frac{\partial h_0}{\partial \zeta}\,\delta\zeta + \left\langle \frac{\partial f_0}{\partial \underset{\sim}{u}}, \delta\underset{\sim}{u} \right\rangle_{\Omega} + \left\langle \underset{\sim}{\lambda}^J \left(\frac{\partial (L\underset{\sim}{z})}{\partial \underset{\sim}{u}} - \frac{\partial Q}{\partial \underset{\sim}{u}}\right), \delta\underset{\sim}{u} \right\rangle_{\Omega} + \left\langle \frac{\partial g_0}{\partial \underset{\sim}{z}}, \delta\underset{\sim}{z} \right\rangle_{\partial\Omega}$$

$$- \left\langle C\underset{\sim}{\lambda}^J, \delta z \right\rangle_{\partial\Omega}$$

$$= \frac{\partial h_0}{\partial \zeta}\,\delta\zeta + \left\langle \left(\frac{\partial f_0}{\partial \underset{\sim}{u}} + \underset{\sim}{\lambda}^J \left(\frac{\partial (Lz)}{\partial u} - \frac{\partial Q}{\partial \underset{\sim}{u}}\right)\right),\ \delta\underset{\sim}{u} \right\rangle_{\Omega}$$

$$= \frac{\partial h_0}{\partial \zeta}\,\delta\zeta + \left\langle \bar{\underset{\sim}{\Lambda}},\ \delta\underset{\sim}{u} \right\rangle_{\Omega}. \tag{23}$$

$$\bar{\underset{\sim}{\Lambda}} = \frac{\partial f_0}{\partial \underset{\sim}{u}} + \underset{\sim}{\lambda}^J \left(\frac{\partial (Lz)}{\partial \underset{\sim}{u}} - \frac{\partial Q}{\partial \underset{\sim}{u}}\right) \tag{24}$$

is the sensitivity vector if $\frac{\delta h_0}{\delta \zeta}\,\delta\zeta \simeq 0$, i.e. the first natural frequency does not change significantly with the admissible changes in the design.

If the term $\frac{\partial h_0}{\partial \zeta}\,\delta\zeta$ can not be ignored, a substitution of (17) into (23) yields

$$\delta J = \frac{\partial h_0}{\partial \zeta} \left\langle M\underset{\sim}{y}_1, \underset{\sim}{y}_1 \right\rangle^{-1} \left\{ \left\langle \left[\frac{\partial (L'\underset{\sim}{y}_1)}{\partial \underset{\sim}{u}}\,\underset{\sim}{y}_1 - \zeta\,\frac{\partial (M\underset{\sim}{y}_1)}{\partial \underset{\sim}{u}}\,\underset{\sim}{y}_1\right],\ \delta\underset{\sim}{u} \right\rangle_{\Omega} \right\}$$

$$+ \left\langle \bar{\underset{\sim}{\Lambda}},\ \delta\underset{\sim}{u} \right\rangle_{\Omega} = \left\langle \underset{\sim}{\Lambda}^J,\ \delta\underset{\sim}{u} \right\rangle_{\Omega}. \tag{25}$$

$\underset{\sim}{\Lambda}^J$ is the sensitivity vector for optimization of J.

It is significant that no boundary terms (i.e. products in H_2) occur in the formula (25).

6. <u>An example</u>. Consider the dynamic problem of thin plate theory represented by the equation

$$\nabla^2(D(x,y)\nabla^2 W) + \rho\frac{\partial^2 W}{dt^2} = Q(x,y)e^{j\omega t}, \ \underset{\sim}{x} \in \Omega, \ t \geq 0.$$

After separation of variables, this is reduced to an eigenvalue problem

$$(L' + \zeta I) \ W(\underset{\sim}{x}) = Q(\underset{\sim}{x}) \qquad \underset{\sim}{x} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{26}$$

where $L' = \nabla^2(D(x,y)\nabla^2)$ and $M \equiv I$ is the identity operator.

Design a plate of a given weight (say unity) such that the average kinetic energy takes an approximately extremal value. The design parameter $u(\underset{\sim}{x})$ is the thickness of the plate. The plate is clamped on the boundary. In fact this problem is equivalent to looking for a low approximate value of the sensitivity vector $\underset{\sim}{\Lambda}$.

The cost functional J is identified as

$$J = \mu_0 \int_\Omega u(\underset{\sim}{x}) \ d\underset{\sim}{x} + k\zeta \tag{27}$$

where k is a known constant. (Recall that ζ is proportional to ω^2). Identifying $h_0(\zeta)$ as $k\zeta$, $g_0 \equiv 0$, $f_0 = \mu_0 u$, the sensitivity vector is computed as

$$\Lambda^1 = \mu_0 + k \left\langle y_1, y_1 \right\rangle^{-1} \ \{\left\langle \nabla^2 \left(\frac{Eu_0^2}{3(1-\nu^2)} \ \nabla^2 y_1 + \frac{Eu_0^3}{12(1-\nu^2)} \ \frac{\partial(\nabla^2 y_1)}{\partial u}\right), y_1 \right\rangle$$

$$- \zeta \left\langle \frac{\partial y_1}{\partial u}, y_1 \right\rangle . \tag{28}$$

Supposing that ζ_1 and $y_1(x)$ are known and $\frac{\partial y_1}{\partial u}$ is computed, the sensitivity vector (28) can be computed in terms of known physical quantities. An optimum design requires near zero sensitivity. In order that u_0 is optimal it is necessary for Λ^J to vanish. However equation (28) is difficult to solve for u_0 and gradual improvement approach is generally used for suboptimal design. If Λ^J is not zero then following a steepest descent appraoch one could choose δu in the direction of $-\Lambda^J$. This is locally optimal direction. In general the constraints improved by physical consideration may not be satisfied by altered design vector $\mu_0 + \delta u$. Various versions of the gradient-projection technique have been used to correct this defect.

For an arbitrarily chosen design, the decrease in the magnitude of the sensitivity vector indicates the degree of improvement in the corresponding step of the steepest descent algorithm.

7. <u>A theoretical foundation of the gradient projection algorithm</u>.

Suppose that one wishes to minimize the cost functional $J(\underset{\sim}{z}(\underset{\sim}{u}),\underset{\sim}{u})$ subject to constraints of the form

$$A\ \underset{\sim}{z}(\underset{\sim}{u}) = \underset{\sim}{b},$$

where $\underset{\sim}{z}$ belongs to a Hilbert space H_1, b to H_2, and A: $H_1 \to H_2$ is a bounded operator with closed range, whose domain is dense

in H_1. For subsequent analysis, N(A) denotes the null space of A and $\underset{\sim}{u}$ denotes an m-dimensional vector, called the design.

It is assumed that $J(\underset{\sim}{z}(\underset{\sim}{u}),\underset{\sim}{u}) = \hat{J}(\underset{\sim}{u})$ is a Frechet differentiable functional. $\hat{J}'(\underset{\sim}{u})$ will denote the gradient of J. In the previous discussion of sensitivity (parts 1 - 5) of this paper $J'(\underset{\sim}{u})$ was identified with the vector Λ^J.

Suppose an initial choice of design $\underset{\sim}{u}$ is made, $\underset{\sim}{u}_0$, with the corresponding value $\hat{J}(\underset{\sim}{u}_0)$ computed. $A\underset{\sim}{z}(\underset{\sim}{u}_0) = \underset{\sim}{b}(\underset{\sim}{u}_0)$ is (approximately) satisfied. Choose $\delta\underset{\sim}{u}$, so that $\|\delta\underset{\sim}{u}\|^2 \leq \varepsilon$ (in fact the choice will be $\|\delta\underset{\sim}{u}\|^2 = \varepsilon$, unless a change or a refinement of step size is required). If no constraints were present, the steepest descent condition would be (for some constant $k > 0$) $\delta\underset{\sim}{u} = -k\hat{J}'(\underset{\sim}{u}_0)$. Note that $\hat{J}'(\underset{\sim}{u})$ is an m-dimensional vector in the space dual to U. Mathematically these spaces are indistinguishable, i.e. isometrically isomorphic, even though the physical dimensions are different. For this reason the symbolism $\delta\underset{\sim}{u} = -k\ J'(\underset{\sim}{u})$ makes sense. Of course in the general case such choice of change in the design $\underset{\sim}{u}$ will violate the constraints.

For the sake of convenience the constraint can be put in the form of a functional equality

$$\psi(\underset{\sim}{u}) = 0, \ \psi: \ U \to \mathbb{R}.$$

For example, $A\ \underset{\sim}{z}(\underset{\sim}{u}) - \underset{\sim}{b}(\underset{\sim}{u}) = \emptyset$ can be replaced by $\psi(\underset{\sim}{u}) = \|A\ \underset{\sim}{z}(\underset{\sim}{u}) - \underset{\sim}{b}(\underset{\sim}{u})\| = 0$.

In discussion of this technique the following problem will

be considered first: The functional $J(\underset{\sim}{u})$ is to be minimized, subject to a constraint $A\underset{\sim}{u} = 0$, where A is linear operator, $A: H_1 \to H_2$.

In this case the algorithm describes a consecutive steepest descent and projection on a constraint surface in a single step. $\hat{J}'(\underset{\sim}{u})$ can be uniquely represented as a sum of a vector in $N(\underset{\sim}{A})$ and a vector in the range of A^*; (Recall that $R(A^*) = N(A)^{\perp}$.)

$$-J'(\underset{\sim}{u}_0) = \underset{\sim}{\gamma}_0 + A^*\eta_0 \tag{7.1}$$

where

$$A\underset{\sim}{\gamma}_0 = 0. \tag{7.2}$$

Hence,

$$0 = -AJ'(\underset{\sim}{u}_0) + AA^*\eta_0 \tag{7.3}$$

Since AA^* is invertible on $(N(A))^{\perp}$ (and $\eta_0 \in (N(A))^{\perp}$), η_0 can be computed as

$$\underset{\sim}{\eta}_0 = -(AA^*)^{-1} A(J'(\underset{\sim}{u}_0)). \tag{7.4}$$

$\underset{\sim}{\eta}_0$ is the orthogonal projection of $-\hat{J}'(\underset{\sim}{u})$ on the constraint hyper-surface $A\underset{\sim}{u} = 0$.

Setting $\underset{\sim}{u}_1 = \underset{\sim}{u}_0 + \alpha\underset{\sim}{\gamma}_0$ completes the first iterative step, where $\alpha + \alpha\gamma_0$ is a constant chosen to satisfy constraint on step size, $0 < \alpha \leq 1$. The iteration proceeds according to the formulas

$$\gamma_n = -[I - A^*(AA^*)^{-1}A]\ \hat{J}'(\underset{\sim}{u}_n) \tag{7.5a}$$

$$\underset{\sim}{u}_{n+1} = \underset{\sim}{u}_n + \alpha\underset{\sim}{\gamma}_n. \tag{7.5b}$$

If the problem has a unique solution, this iterative technique converges to that solution (See [12]).

Critique. In most structural design problems, operator A is not linear, A* is generally undefined, and the above technique appears to be difficult to apply. However if the constraint functional is Fréchet differentiable, then locally (for small variations of the parameter vector $\underset{\sim}{u}$) the constraint can be approximated by a linear one. That is the hypersurface $\psi = 0$ can be locally approximated by a supporting hyperplane. However the steepest descent step is projected on such a hyperplane and not on the constraint surface, which means that the constraint is not satisfied, but is only approximately satisfied.

An additional correction could be applied at each step (or only at certain steps) of this iteration to insure exact complience with the constraints. For example, a step $\delta\underset{\sim}{u} = -\beta\psi'$, $0 \leq |\beta| \leq 1$ can be put in, with $|\beta|$ estimated to be $\|\psi\| \cdot \|\underset{\sim}{\psi}'\|^{-2}$, to reduce close to zero the value of $\|\psi\|$. β is positive if ψ is positive, and negative if ψ is negative.

Remark: In most engineering problems exact complience with constraints is not very important, since the constraint conditions are inaccurate and were derived following some simplifying assumptions.

$7^{\underline{a}}$. Haug's variant of the steepest descent algorithm for structural design optimization

In [3] Haug has employed a technique which can be regarded as a variant of the gradient projection method. Further refinements and some changes in this approach were given by Haug, Arora and Matsui in [14]. The problem involves minimization of a functional $J = \psi_o(u,z,\zeta)$, subject to inequality constraints $\phi(u) \leq 0$, $\psi_j(u) \leq 0$ $j = 1,2,\ldots,k$; as in (8) pointwise constraints being replaced by functional constraints. As usual

$$\delta\phi = \langle \phi_u, \delta u \rangle\ ,\ \delta\psi = \langle \psi_u, \delta u \rangle + \langle \psi_z, \delta z \rangle + \langle \psi_\zeta, \delta\zeta \rangle \tag{7.6}$$

Explicit dependence of $\delta\psi$ on δz and $\delta\zeta$ is eliminated by perturbing the state equations (1) and the corresponding Hamiltonian functional:

$$FL(\underset{\sim}{z},\underset{\sim}{u},\underset{\sim}{\lambda}) = \langle L(u)\underset{\sim}{z},\ \underset{\sim}{\lambda} \rangle = \langle Q(\underset{\sim}{x},\underset{\sim}{u}),\ \underset{\sim}{\lambda} \rangle = \ \underset{\sim}{\psi}(u,z), \tag{7.7}$$

where λ is physically a dual element of $Q(x,\underset{\sim}{u})$. Since the dual of H_3 is H_3, $\underset{\sim}{\lambda}$ is an element of the Hilbert space H_3. Taking a Fréchet derivative of FL with respect to $\underset{\sim}{\lambda}$ one recovers the equation (1). Differentiating with respect to z, one arrives at the adjoint system

$$L^*(u)\underset{\sim}{\lambda} = -\underset{\sim}{\psi}_z \tag{7.8}$$

This leads to identities

$$\langle \underset{\sim}{\lambda},\ L\delta \underset{\sim}{z} \rangle + \langle \underset{\sim}{\lambda}, (Lz)_u\ \delta \underset{\sim}{u} \rangle = 0 \tag{7.9}$$

$$\langle L^*\underset{\sim}{\lambda},\ \delta \underset{\sim}{z} \rangle + \langle \underset{\sim}{\lambda},\ (Lz)_u\ \delta \underset{\sim}{u} \rangle = 0 \tag{7.10}$$

$$-\langle \psi_z,\ \delta \underset{\sim}{z} \rangle + \langle \underset{\sim}{\lambda},\ (Lz)_u\ \delta \underset{\sim}{u} \rangle = 0 \tag{7.11}$$

Since $(Lz)_u\ \delta u = -L\ \delta z$, dependence of $\delta\psi$ on δz is eliminated. If $\underset{\sim}{y}$ is an eigenfunction of the system

$$L(u)y = \zeta \, M(u)y \quad , \tag{7.12}$$

where as before M is positive, definite, and symetric, one derives an identity

$$\langle y, \left((Ly)_u - \zeta (My)_u\right) \delta u \rangle = \delta\zeta \quad \text{Since } \langle y, M(u)y \rangle = 1$$

$$\delta\psi = \langle \psi_u, \delta u \rangle + \langle \underset{\sim}{\lambda}, (Lz)_u \, \delta u \rangle + \psi_\zeta \langle y, ((Ly)_u - \zeta (My)_u \, \delta u \rangle$$

$$= \langle \underset{\sim}{\lambda}, \delta u \rangle \leq \Delta \underset{\sim}{\psi} \, . \tag{7.13}$$

Here one is approximating the non-linear terms by first order linear approximations.

A constraint on the magnitude of δu is imposed by requiring $\langle W\delta u, \delta u \rangle \leq \xi^2$ where the operator W (the weighing operator) is positive definite, and symetric. ξ^2 is chosen sufficiently small for the particular problem.

Application of the multiplier rule leads to the set of equations

$$\lambda^o + \mu \cdot \phi_u + \gamma \cdot \underset{\sim}{\lambda} + 2\nu \cdot W \, \delta \underset{\sim}{u} = 0$$

$$\phi_u \cdot \delta \underset{\sim}{u} - \Delta \underset{\sim}{\phi} = 0 \tag{7.14}$$

$$\gamma \cdot \{\langle \underset{\sim}{\lambda}, \delta \underset{\sim}{u} \rangle - \Delta \underset{\sim}{\psi}\} = 0 \tag{7.15}$$

$$\nu \cdot \{\langle W\delta u, \delta u \rangle - \xi^2\} = 0 \tag{7.16}$$

Here (.) means the product $A \cdot B = \sum_{i=1}^{n} A_i \cdot B_i$ as used in the system of equations (7.14) - (7.16). The equations (7.7) - (7.11) can be formally solved for $\delta \underset{\sim}{u}$, while (7.14) - (7.16) can be used to eliminate the multipliers.

The following formulas are derived.

$$\delta u^1 = W^{-1}\left[I - \tilde{\phi}_u^T \left(\tilde{\phi}_u W^{-1} \tilde{\phi}_u^T\right)^{-1} \tilde{\phi}_u W^{-1}\right]$$

$$\times \left[\Lambda^0 - \tilde{\Lambda} \, M_{\psi\psi}^{-1} \, M_{\psi j}\right]$$

$$\delta u^2 = W^{-1}\left[I - \tilde{\phi}_u^T\left(\tilde{\phi}_u W^{-1}\tilde{\phi}_u^T\right)^{-1}\tilde{\phi}_u W^{-1}\right]$$

$$\times\left[\tilde{\Lambda}\, M_{\psi\psi}^{-1}\left(\Delta\psi - M_{\psi\phi}\right)\right] + W^{-1}\tilde{\phi}_u^T\left(\tilde{\phi}_u W^{-1}\tilde{\phi}_u^T\right)^{-1}\Delta\tilde{\phi},$$

where

$$M_{\psi j} = \left\langle\tilde{\Lambda}, W^{-1}\left[I - \tilde{\phi}_u^T\left(\tilde{\phi}_u W^{-1}\tilde{\phi}_u^T\right)^{-1}\tilde{\phi}_u W^{-1}\right]\Lambda^0\right\rangle,$$

$$M_{\psi\phi} = \left\langle\tilde{\Lambda}, W^{-1}\tilde{\phi}_u^T\left(\tilde{\phi}_u W^{-1}\tilde{\phi}_u^T\right)^{-1}\Delta\tilde{\phi}\right\rangle,$$

and

$$M_{\psi\psi} = \left\langle\tilde{\Lambda}, W^{-1}\left[I - \tilde{\phi}_u^T\left(\tilde{\phi}_u W^{-1}\tilde{\phi}_u^T\right)^{-1}\tilde{\phi}_u W^{-1}\right]\tilde{\Lambda}\right\rangle.$$

$$M_{jj} = \left\langle\Lambda^0, W^{-1}\left[I - \tilde{\phi}_u^T\left(\tilde{\phi}_u W^{-1}\tilde{\phi}_u^T\right)^{1}\tilde{\phi}_u W^{-1}\right]\Lambda^0\right\rangle,$$

while δu is given by

$$\delta u = -\frac{1}{2\nu}\,\delta u^1 + \delta u^2,$$

δu^2 is basically the constraint correction, δu^1 gives a step in the direction of negative gradient projection. Simultaneous application of δu^1, δu^2 constitutes Haug's version of the modified gradient projection method.

Applications to truss, beam, and plate design can be found in [14].

8. Optimal design of statically determinate structures.

In the statically determinate case, the loads and moments transmitted by the members are independent of the geometric design of each member. Hence if the design vector influences only the cross-section property of each member, the sensitivity vector is relatively easy to compute.

As an example, consider a statically determinate beam in pure bending. The deflection is given by the Betti formula

$$W(x) = \int_0^{\ell} \frac{M(\xi)m(x-\xi)}{EI(\underset{\sim}{u}(\xi))} d\xi = \frac{1}{E} (\frac{M}{I} * m). \tag{8.1}$$

where M (the moment applied) and m (the moment at x due to a unit load at ξ) are independent of the design $\underset{\sim}{u}(\xi)$. Using rules of Frechet differentiation the sensitivity operator $\frac{\partial W}{\partial u}$ is computed as

$$\frac{\partial W}{\partial \underset{\sim}{u}} = \frac{1}{E} ((MI^{-2}) * m) \frac{\partial I}{\partial u} \tag{8.2}$$

In the statically determinate case it is therefore unnecessary to eliminate $\frac{\partial W}{\partial u}$, since all terms in the formula (8.2) above are known.

The sensitivity vector Λ^J can be identified with the Fréchet derivative

$$\frac{1}{E} (MI^{-2}(u) * m) \frac{\partial I(u)}{\partial u},$$

if no constraints are imposed. A trivial result is obtained by setting $\Lambda^J = 0$.

If a weight constraint, and or minimum cross-section, or maximum stress constraint is imposed, the optimum design solution becomes non-trivial.

In case where a functional relation $\psi(\underset{\sim}{u})$ can be derived for the constraint $\psi \leq c$, the necessary condition for optimization of the design $\underset{\sim}{u}$ is of the form

$$\frac{\lambda_o}{E} \quad (MI^{-2}(u) \; * \; m) \; \frac{\partial I(u)}{\partial u} \; + \lambda_1 \frac{\partial \psi}{\partial u} = 0.$$

This remark is illustrated by the following design problem.

9. <u>A design problem for an optimal shape of a gun barrel</u>.

The gun tube design can be regarded as a one parameter family of maps $h(x) \varepsilon H_1 \rightarrow \max |\theta(x)| \varepsilon R$. H_1 is a class of functions obeying certain constraints, and it will be called the class of admissible thickness design functions. Physically $h(x)$, $x \varepsilon [0,\ell]$, represents the thickness of the gun tube. R_0 will denote the caliber.

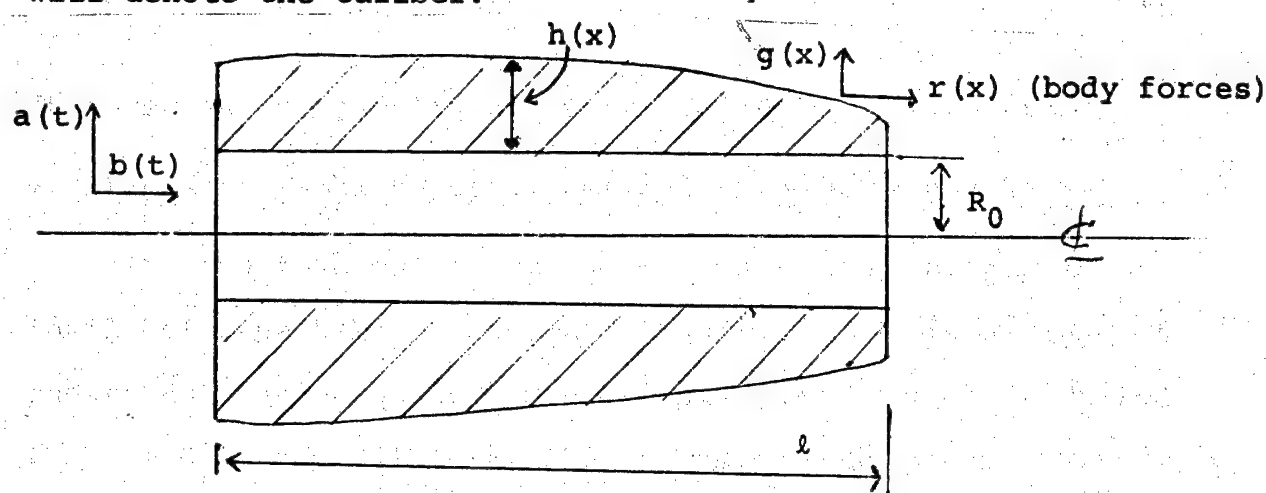


figure 1

We formulate the optimization problem as follows:

Find an $L_2[0,\ell]$ function $\tilde{h}(x)$, $x \in [0,\ell]$ such that $\min\limits_{h} (\max\limits_{x} |y'(\tilde{h}(x),x)|) \leq \min\limits_{h}(\max\limits_{x}|y'(h(x),x)|)$ for any other choice of $h(x) \in L_2[0,\ell]$, with the constraint $h(x) \leq h_0$ $\forall x \in [0,\ell]$, the maximum stress constraint $\sigma_{max} \leq \sigma_0$, and weight constraint $W \leq W_0$. The gun tube is assumed to be a cantilevered beam subjected to the a priori given accelerations in the direction of its axis and in the plane perpendicular to its axis. If more than one optimal design exists, then the design which is optimal according to our criterion (i.e. it minimizes max $\frac{dy(h(x),x)}{dx}$) and which also results in smallest total weight is deemed the optimal design.

Mathematical formulation of the problem

Since the beam is simply cantilevered, and we have a statically determinate case, we can use Prager's approach. The bending moment $M(x)$ and the moment $m(x-x_0)$ (at x due to a unit load positioned at x_0) are independent of the design. Castigliano's theorem can be rewritten in the form:

$$y(x_0) = E^{-1} \int_0^{\ell} (I(x))^{-1} M(x) m(x-x_0) dx. \tag{9.1}$$

We seek to minimize $\max|h'(x_0)| = \max \{ \int_0^{\ell} (I(x)^{-1} M(x)\ m'(x-x_0) dx\}$, $x_0 \in [0,\ell]$, subject to constraints $\int \rho\ dx \leq W_0$ (9.2)

$$h(x) \geq h_0, \quad \left|\frac{\rho\ a_{max}}{A(x)}\right| + \left|\frac{M(x) \cdot C(x)}{I(x)}\right| \leq \sigma_0\ , \tag{9.3}$$

where $A(x) = \pi(R_0 + h(x))^2 - \pi R_0^2$ (9.4[a])

$$I(x) = \frac{\pi\rho(R_0 + h(x))^4}{4} - \frac{\rho\pi R_0^4}{4} \qquad (9.4^b)$$

ρ is the material density, a_{max} is maximum acceleration experienced by the gun mount.

The maximum force (per unit length) due to inertial load is

$$q(x) = -a_{max}\pi\rho(h + R_0)^2 - R_0^2). \qquad (9.5)$$

Note:

In this computation we shall ignore the possibility of torsional vibration and stresses and use the formula $\sigma = \frac{F}{A} \pm \frac{MC}{I}$. It is done for reasons of simplicity. However, incorporating the effects of torsion stress into this computation is quite straightforward.

The static deflection is given by the formula:

$$y(x_0) = E^{-1} \int_o^{\ell} \{\frac{\pi\rho}{4} ([R_0+h(x)]^4-R_0^4)\}^{-1} \cdot M(x) \cdot m(x-x_0)dx. \qquad (9.1^a)$$

It can be shown that:

$$\left.\frac{dy}{dx}\right|_{x=x_0} = E^{-1} \int_o^{\ell} \{\frac{\pi\rho}{4} ([R_0+h]^4-R_0^4\}^{-1} M(x) \frac{d}{dx_0} (m(x-x_0)) \, dx \qquad (9.6)$$

Let us denote the quantity (9.6) by $\theta(x_0)$. The problem is now reduced to the finding of necessary conditions for minimization of $\max_{x_0} |\theta(x_0)|$ subject to constraints (9.3).

The formulas for $M(x)$ and $m(\xi)$ $(\xi=x-x_0)$ are given respectively by: $M(x) = \int_x^{\ell} q(\xi) \cdot (\xi-x)d\xi$ $\qquad (9.5^a)$

where $q(x)$ is given by equation (9.5)

$m(\xi) = U(-\xi)$, where $U(\xi)$ is the unit step function

$$U(\xi) \quad = 0 \text{ if } \xi < 0$$
$$\qquad = 1 \text{ if } \xi > 0.$$

$U'(\xi) = \delta(\xi)$ (δ will denote the Dirac delta function).

Hence we have

$$\theta(x) = E^{-1} \left(M(x) \frac{\pi\rho}{4} [(R_0+h)^4 - R_0^4]\right)^{-1} * \delta(x)$$

$$= \left(\frac{\pi E \rho}{4}\right)^{-1} M(x) \cdot ([R_0+h(x)]^4 - R_0^4). \qquad (9.6^a)$$

(* denotes the operation of convolution).

We consider two possibilities:

(a) $\max\limits_{x \in [0,\ell]} |\theta(x)|$ occurs at an interior point of $[0,\ell]$.

(b) $|\theta(\ell)| = \max\limits_{x\in[0,\ell]} |\theta(x)|$.

The case (b) will be ignored for the time being, since $\theta(\ell)$ is easily computed (for a given design) and Prager's arguments ([3]) can be duplicated with only minor modifications. The case (a) is more complex.

Our discussion will concentrate on minimization criteria for the case (a), and only a final comparison shall be made to check if $|\theta(\ell)|$ exceeds $\max\limits_{x\in[0,b]} |\theta(x)|$, $b < \ell$.

<u>A necessary criterion for optimization of an interior maximum (case (b)).</u>

Following the Komkov-Coleman technique outlined in [4] we equate to zero the Fréchet derivative of the functional:

$$\Phi(h) = \lambda_0 \theta(h) + \lambda_1 (h-h_0) + \lambda_2 \{\frac{\rho|a_{max}|}{A(h)} + \frac{|M(h)|c(h)}{I(h)} - \sigma_0\}$$

$$+ \lambda_3 \frac{d}{dx} (\theta(h)) + \lambda_4 (\int_0^{\ell} A(h(x))dx - W_0). \qquad (9.7)$$

$$\Phi_h = \lambda_0 (\frac{\pi E\rho}{4})^{-1} \{((x^*(-2\pi\rho h)) \ [(R_0+h)^4 - R_0^4] - (\pi\rho x^*((h+R_0)^2 - R_0^2))$$

$$4(R+h)^3 + \lambda_1 + \lambda_4 A_h + \lambda_2 (\mathrm{sign}(M) \cdot (\frac{MC}{I})_h) + \lambda_3 \frac{d}{dx} (\theta_h) = 0 \qquad (9.8)$$

The Frechet derivative of $\theta(h(x),x)$ denoted by θ_h is exactly the first term of the above expression, i.e.

$$\theta_h(x) = (\frac{\pi E\rho}{4})^{-1} \{((x^*(-2\pi\rho h)) \cdot [(R_0+h)^4 - R_0^4] + (-\pi\rho x^*$$

$$((h + R_0)^2 - R_0^2) \cdot 4(R_0+h)^3\}. \qquad (9.8^a)$$

(as before * denotes the operation of convolution). $\lambda_4, \lambda_1, \lambda_2$ obey the inequalities:

$$\lambda_1 \leq 0, \ \lambda_2 \leq 0, \ \lambda_4 \leq 0. \qquad (9.9)$$

λ_0, λ_1, λ_2, λ_4, are constants, but λ_3 must be regarded as an

unknown function. The relation (9.8) and inequalities (9.9) determine a necessary condition for a maximum of $\theta(x)$ subject to the constraints (9.3).

The necessary condition for optimization of $\max|\theta(x)|$ (x-an interior point of $[0,\ell]$) given by the relation (9.8) states that $\frac{d}{dx}(\theta_h)$ lies in the forward cone of the three-dimensional subspace of $L_2[0,\ell]$ spanned by θ_h, A_h, and by

$$\{(\text{sign } M(h))\cdot(\frac{Mc}{I})_h - \frac{\rho\ a_{max}}{A^2(h)} A_h\}, \tag{9.10}$$

where subscripts denote Fréchet differentiation. $\frac{d}{dx}(\theta_h)$ is computed directly.

$$\frac{d}{dx}\theta_h = (\frac{\pi E\rho}{4})^{-1} \{[-2\pi\rho h(R_0+h)^4-R_0^4] + [(x*(-2\pi\rho h))\cdot((R_0+h)^4-R_0^4]$$

$$-(4\pi\rho*((h+R_0)^2-R^2))\cdot(R_0+h)^3] - 3(\pi\rho x*(h+R_0)^2-R_0^2)\cdot$$

$$(R_0+h)^2\ h'(x)\}\ . \tag{9.11}$$

We observe that $\theta_h(x)$ satisfies a first order differential equation (8), which is;

$$\lambda_0\theta_h + \lambda_3 \frac{d}{dx}(\theta_h) = -\lambda_1-\lambda_2\ (\text{sign } M(h)\ (\frac{Mc}{I})_h)-\lambda_4 A_h. \tag{9.12}$$

$\lambda_3 \equiv 0$ corresponds to the case when maximum of $\theta(x)$ occurs at the boundary point $x = \ell$.

Moreover, $\lambda_3(\bar{x}) = 0$ for some $\bar{x} \varepsilon [0,\ell]$ implies that $\bar{x}$ is a singular point of the equation (8), and;

$$\theta_h(\bar{x}) = \frac{-\lambda_1}{\lambda_0} - \frac{\lambda_2}{\lambda_0} (\text{sign } M(h(\bar{x}))\cdot(\frac{M(h)\ (R_0+h)}{I(h)})_h \Big|_{x=\bar{x}} - \frac{\lambda_4}{\lambda_0} A_h \Big|_{x=\bar{x}} \tag{9.13}$$

However, θ_h is a known function of h(x) (see equation 8a) and the numbers obtained by computation (9.13) and (9.8a) after substitution of some design function h(x) will generally fail to be equal. The possibility that they could be equal will however remain, and an additional check will be added to the computational algorithm to take care of that possibility.

The preceeding example could be rewritten to incorporate the results of sections 4 and 5. Assuming that the gun barrell vibrates with a natural frequency ζ_i, the discussion above can be modified by incorporating the additional term similar to the second term in equation (28), where ∇^2 is replaced by $\frac{d^2}{dx^2}$, and Y_1 by $Y_i(x)$ which is corresponding eigenshape.

10. An example of computation

Using the theoretical approach of sections 4 and 5. E.T. Haug, S.S. Arora and K. Matsui have programmed an optimization algorithm for a simply supported beam of rectangular cross-section with width/height ratio of 1:2, and $I(x) = \frac{1}{6}u^2(x)$ where u(x) is the design parameter.

The loading is shown below

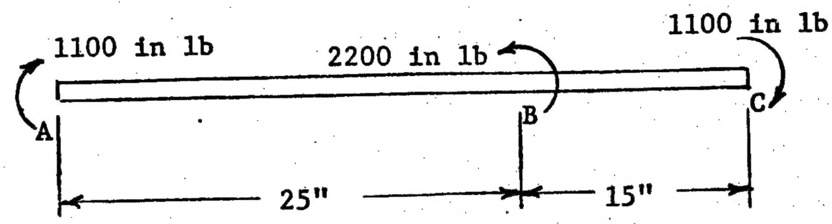
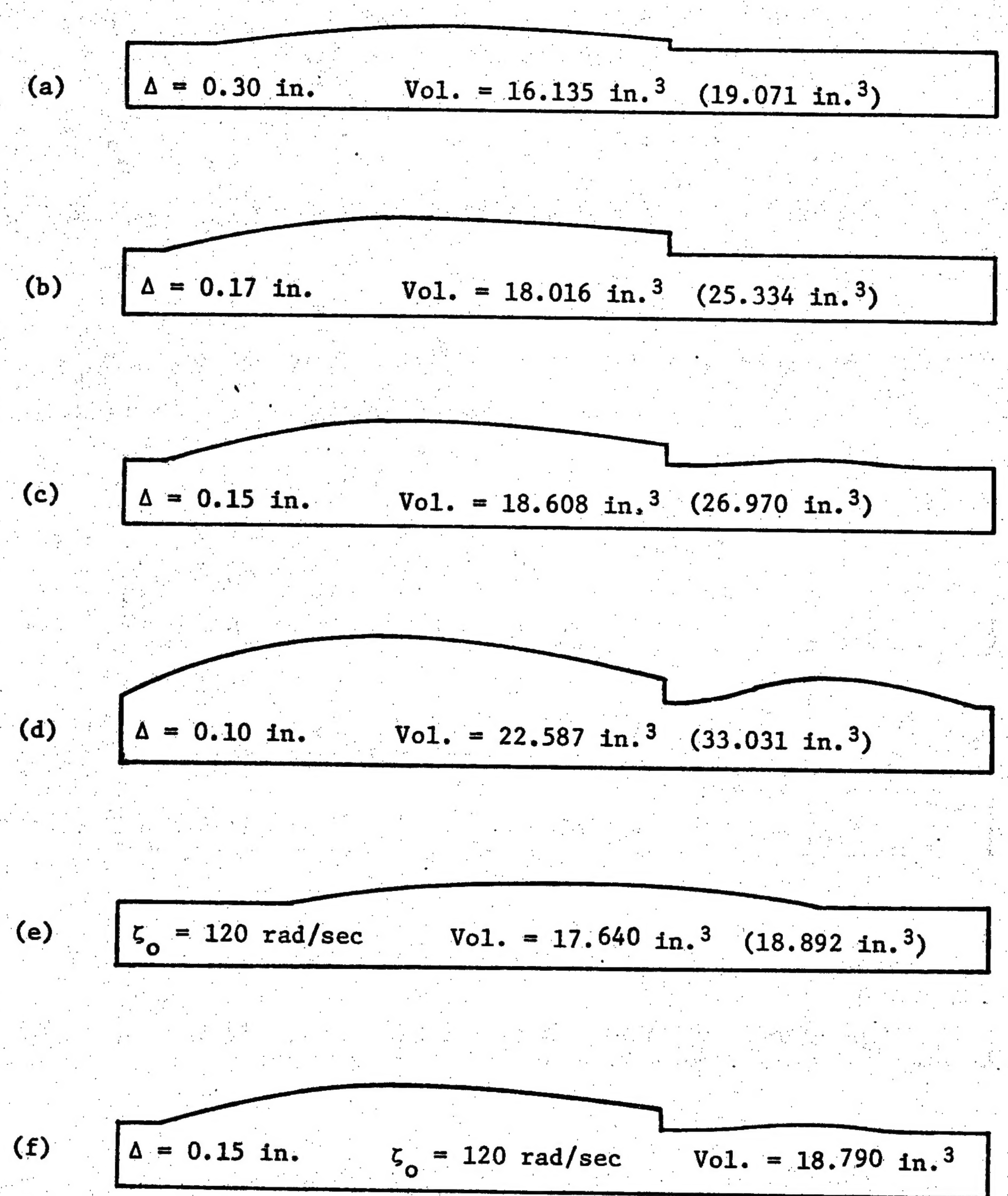


Figure 2:

The lower bound on cross-section was selected to be .3645 in 2. The displacement constraint is Δ, the first natural frequency is denoted by ζ_0. The cost functional is the total volume of material. (Here $E = 10^7$ psi, $\rho = .00208$ slugs/in^3. The computation was performed on IBM 360 at the University of Iowa. Convergence was rapid-no more than 35 iterations were required in any computation. The optimum design is illustrated below for various values of Δ.

It is interesting to observe how complexity of design is influenced by the choice of Δ_0 in the constraint $\Delta \leq \Delta_0$.

A similar phenomenon is observed if constraint on the value of maximum stress is imposed, instead of the maximal displacement.



Variation of Cross-Sectional Area

11. Conclusions.

The technique derived in sections 4 - 6, and the more specialized results of sections 8 and 9 have been shown to be adaptable to numerical optimization schemes which compare favorably with other methods, such as adaptation of Pontryagin's maximality principle proposed in [3]. Direct solutions of the design problem with complex, non-linear constraints represents a new result in this area of engineering analysis. Even the optimization problem discussed in section 8, and illustrated in example 9 can not be derived by a direct application of Prager's result. The problem discussed there is not the dual of Prager's problem [7], nor can it be deduced as a consequence of the works of Prager's collaborators (which have not been quoted in the references) to best of our knowledge.

Computing experience with the approach suggested in this paper have been satisfactory. In fact satisfactory numerical results have been obtained in cases where theoretically the technique derived here is not justifiable.

Additional theoretical work is clearly needed. Perhaps, a more sophisticated approach based on abstract variational principles (see [13]) may by-pass some of the difficulties encountered in justifying purturbation type arguments, or arguments based on vanishing of Fréchet derivative and the use of Lagrangian multiplier rule in problems of structural mechanics.

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